Proper Orthogonal Decomposition for Analysis of High-Power Virtual Cathode Oscillations

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***Abstract*—The Proper Orthogonal Decomposition (POD) is a mode decomposition technique that can be used for the creation of reduced order models or for analysis of complex dynamic systems. In this work, we apply the POD to analyze the characteristics of the oscillations present in an electron beam with formation of an oscillating virtual cathode.**

I. Introduction

Proper Orthogonal Decomposition (POD) is a model order reduction technique that extracts the spatiotemporal behavior from a problem of interest [1]–[5]. This spatiotemporal behavior is represented by a set of coupled spatial and temporal modes [6], [7], which makes POD especially suitable for analysis and applications in nonlinear dynamic systems. POD has been used for creation of reduced-order models [8]–[10]. In particular, reference [11] applies POD in the context on electromagnetic particle-in-cell (PIC) simulations of kinetic plasmas [12], [13]. Here, POD is explored as an efficient tool to analyze the characteristics of coherent radiation generated by an oscillating virtual cathode inside a cavity.

II. Proper Orthogonal Decomposition

The POD allows any dynamic function of interest to be modelled as:

$$\mathbf{f}(\mathbf{x},t) \approx \sum_{i=1}^{N_m} \sigma_i \Psi_i(\mathbf{x}) \Phi_i(t), \tag{1}$$

where N_m is the total number of POD modes, σ_i is the modal amplitude, Ψ_i are (in general vector-valued) spatial modes and Φ_i are (scalar-valued) temporal modes. This representation is obtained in numerical fashion by arranging sampled data from $\mathbf{f}(\mathbf{x},t)$ (which may come from experimental data or from "full-order" numerical simulations) in a snapshot matrix $[\mathbf{A}]$ such that its column space spans the spatial variation of the sampled function and its row space spans the temporal variation, i.e., each column of $[\mathbf{A}]$ consists of the values of $\mathbf{f}(\mathbf{x},t)$ at different spatial points, at a given time, in the domain of interest. The snapshot matrix can then be decomposed via singular value decomposition (SVD) as $[\mathbf{A}] = [\mathbf{U}] \cdot [\mathbf{\Sigma}] \cdot [\mathbf{V}]^T$. The columns of $[\mathbf{U}]$ provide a basis for the column space of $[\mathbf{A}]$, hence for the spatial distribution of $\mathbf{f}(\mathbf{x},t)$, so they are the spatial modes Ψ_i. The columns of $[\mathbf{V}]$ provide a basis for the row space of $[A]$, hence for the temporal evolution of $\mathbf{f}(\mathbf{x},t)$, so they are the temporal modes Φ_i. The singular values are the spatiotemporal amplitudes σ_i.

III. Virtual Cathode Formation

Virtual cathode devices are well-known to generate high-power coherent radiation [14], [15]. As electrons travel inside a waveguide or cavity, their kinetic energy is transferred into a space-charge potential. If the electron injection rate is high enough, this potential increases in magnitude to the point where a virtual cathode forms inside the beam, i.e. a region of space with potential equal to or greater than that of the emitting cathode. This causes electrons inside the beam to be deflected back towards the direction they were injected from, and can give rise to oscillations that generate coherent radiation.

IV. Application and Discussion

In the example that follows, all data is generated by an exactly charge-conserving finite-element-based PIC algorithm, the details of which can be found in [12], [16], [17]. Consider a two-dimensional square cavity with perfect electric conductor walls measuring $l = 10$ [mm]. The cavity is initially without any particles and with zero electromagnetic fields. We inject electrons at the bottom wall of the cavity with an initial velocity of $\mathbf{v} = \hat{\mathbf{y}}\ 0.33c$, and they are absorbed upon hitting the upper wall. The injection rate is set such that the electric current exceeds the space charge limit, causing an oscillating virtual cathode to emerge. We stress that there are no external fields or sources in the simulation, in other words the behavior is entirely driven by the self-interactions of the electrons and the coupling of the self-fields with the cavity walls.

Fig. 1 shows the average position, along the y axis, of all particles inside the domain. It can be seen that the beam initially travels unimpeded but then gets partially reflected and starts oscillating around the position $y_0 = 1$ [mm]. Note that this is *not* the position of the virtual cathode, but rather the average position of all electrons in the domain, including those that escape the cathode and continuing flowing towards the upper boundary of the computational domain (as illustrated in the inset figure).

Fig. 2 shows the first three spatial electric field modes obtained from the POD technique in this case. The blue arrows denote the direction of the electric field, while the background colormap denotes the normalized field intensity. The first mode represents the steady beam behavior that is dominant while the injection current is below the space-charge limit (see [11] for details). The second and third mode represent coherent cavity modes generated by the virtual cathode oscillation.

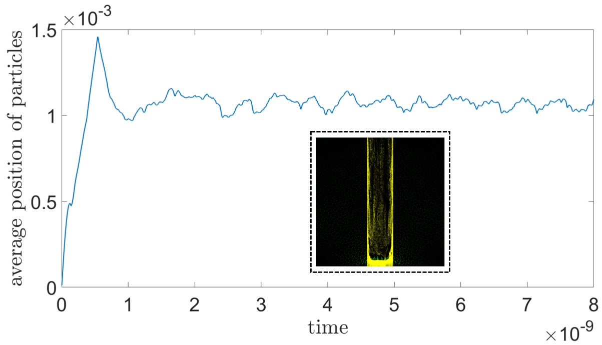
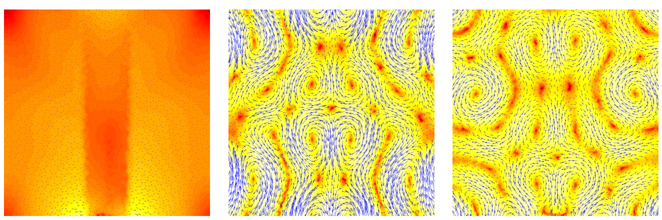


Fig. 1. Average position, on the y-axis, of all the particles inside the domain. The beam initially travels unimpeded in the positive direction, but then gets reflected and starts oscillating around the position $y_0 = 1$ [mm]. The inset shows the physical configuration of electrons at an arbitrary time; notice how most of the electrons get stuck in the bottom of the domain due to the virtual cathode, with the exception of the side streams common in 2D configurations and a few electrons that stream past the V.C.



(a) First mode. (b) Second mode. (c) Third mode.

Fig. 2. Spatial distributions of the first three electric field modes. The first mode is a simple straight beam mode reminiscent of when the injected current is below the space charge limit, while the second and third modes display cavity configurations induced by the coherent oscillation of the electrons.

Fig. 3 shows the associated temporal modes, multiplied by their respective modal amplitudes, for comparison. It can be clearly seen that the temporal mode associated with the first (steady beam) mode is constant trough time, but the temporal behavior associated with the second mode is increasing with time, while the third mode appears as an oscillating correction to the overall dynamics.

This example showcases how the complex phenomena behind coherent radiation from a virtual cathode can be efficiently decomposed into simple modes by the POD technique. This allows for a more insighful analysis of the physical problem by a sparse reduced-order model that still captures the relevant physics of the problem. Bear in mind that this is not an actual or optimal device design, but merely an illustrative example for the POD methodology.

ACKNOWLEDGEMENT

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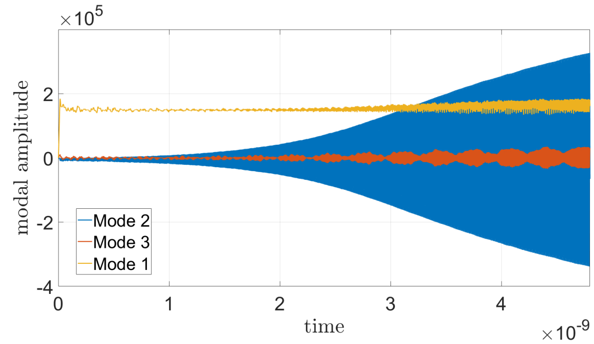

Fig. 3. Temporal evolution of the first three modes for the electric field. The first mode displays a steady, constant behavior, while the second mode has oscillatory growth and the third mode displays steady oscillation at a smaller amplitude.

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