The Degeneracy of the Dominant Mode in Rectangular Waveguide

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***Abstract*—We explore the exceptional point of degeneracy (EPD) at the cutoff of the dominant mode in a uniform rectangular waveguide. We show that the system matrix describing the wave propagation is similar to a Jordan matrix, and we look at the field at the degeneracy point that exhibits algebraic growth along the waveguide as a result of the Jordan matrix description. Field expressions for the dominant mode are derived at the EPD. A numerical example is used to show the degeneracy and its associated field behavior.**

***Index Terms*—Rectangular waveguides, degeneracies, Jordan matrix, dominant-mode, algebraic growth.**

I. Introduction

EXCEPTIONAL points of degeneracy (EPDs) are points in parameters space where two or more eigenmodes of a waveguide coalesce into one single eigenmode. The dispersion relation for structures exhibiting an EPD of order n has the behavior $(\omega-\omega_e) \propto (k-k_e)^n$ in proximity of the EPD, where ω and k are the angular frequency and the modal wavenumber, respectively, and the EPD is designated with the superscript e. Such dispersion behavior is accompanied by severe reduction in the group velocity of waves propagating in those structures [1] and giant increase in the loaded quality factor of the structure [2].

The conditions that lead to EPDs are still ambiguous in some structures and the link between EPDs and the accompanied system properties are still not completely determined. In uniform (i.e., z-invariant) waveguide EPD occurs when two modes, propagating in opposite directions, coalesce at $k = 0$, i.e., the cut-off frequency. In this paper, we revisit the cutoff condition under the light of EPDs and show some important features and concepts associated to it. We show the degeneracy of the dominant mode in uniform rectangular waveguides. We demonstrate that the system matrix describing modal propagation at the EPD is similar to a Jordan matrix and show that EPD is branch point. We also prove that the fields have algebraic growth along the waveguide when it operates at an EPD.

II. Formulation

Consider a *uniform* rectangular waveguide with cross-section dimensions $a \times b$ as in Fig. 1(a). Starting with Maxwell's equations for time-harmonic fields in the waveguide

$$\begin{aligned} \nabla \times \mathbf{E} &= -j\omega\mu\mathbf{H} \\ \nabla \times \mathbf{H} &= j\omega\varepsilon\mathbf{E} \end{aligned} \tag{1}$$

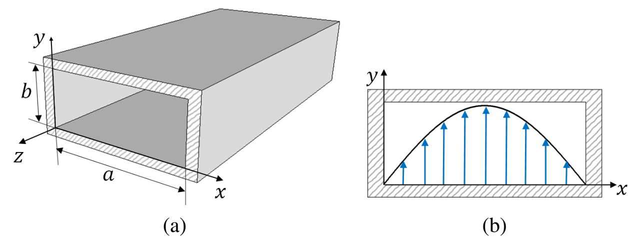

Fig. 1: Uniform rectangular waveguide with cross section dimensions $a \times b$: (a) perspective view and (b) electric field mode profile of the dominant mode TE_{10}.

and then using separation of variables [3] the fields of the waveguide dominant TE_{10} mode are written as

$$\begin{aligned} E_y(x,y,z) &= \sin(k_{x0}x)\, e_y(z) \\ H_x(x,y,z) &= \sin(k_{x0}x)\, h_x(z) \\ H_z(x,y,z) &= \cos(k_{x0}x)\, h_z(z) \end{aligned} \tag{2}$$

where $k_{x0} = \pi/a$. Substituting (2) in (1), we have

$$\begin{aligned} \frac{de_y}{dz} &= j\omega\mu h_x, \\ \frac{dh_x}{dz} &= j\left(\frac{k_0^2 - k_{x0}^2}{\omega\mu}\right) e_y, \end{aligned} \tag{3}$$

where $k_0 = \omega\sqrt{\mu\varepsilon}$. For convenience, we define the two-dimensional state vector $\boldsymbol{\Psi}(z) = [\, e_y(z) \quad h_x(z) \,]^T$ that comprises transverse electric and magnetic fields at a coordinate z. Thus, we write (3) in first order differential equation form as

$$\frac{d\boldsymbol{\Psi}(z)}{dz} = -j\underline{\mathbf{M}}\boldsymbol{\Psi}(z), \tag{4}$$

where $\underline{\mathbf{M}}$ is a 2×2 system matrix given by

$$\underline{\mathbf{M}} = \begin{pmatrix} 0 & -\omega\mu \\ \dfrac{k_{x0}^2 - k_o^2}{\omega\mu} & 0 \end{pmatrix}. \tag{5}$$

The solutions of (4) assuming a state vector of the kind $\boldsymbol{\Psi}(z) \propto e^{-jkz}$ is found by solving the eigenvalue problem

$$\underline{\mathbf{M}}\boldsymbol{\Psi}(z) = k\boldsymbol{\Psi}(z). \tag{6}$$

The dispersion relation of the mode can be determined as $D(k) = \det(\mathbf{M} - k\mathbf{I}) = k_0^2 - k_{x0}^2 - k^2$. Therefore, the two eigenvalues k_1 and k_2 and their corresponding eigenvectors $\boldsymbol{\Psi}_1$ and $\boldsymbol{\Psi}_2$ of the eigenvalue problem in (6) are

$$k_1 = \sqrt{k_0^2 - k_{x0}^2}\ , \quad k_2 = -\sqrt{k_0^2 - k_{x0}^2}, \tag{7}$$

$$\mathbf{\Psi}_1 = \left(\begin{array}{c} 1 \\ \frac{-\sqrt{k_0^2 - k_{x0}^2}}{\omega\mu} \end{array} \right), \quad \mathbf{\Psi}_2 = \left(\begin{array}{c} 1 \\ \frac{\sqrt{k_0^2 - k_{x0}^2}}{\omega\mu} \end{array} \right). \tag{8}$$

Degeneracy occurs when two eigenmodes coalesce in their eigenvalue and eigenvectors as $k_1 = k_2$ and $\mathbf{\Psi}_1 = \mathbf{\Psi}_2$. This happens when the rectangular waveguide is operating at the cutoff frequency, i.e., $k_0 = k_{x0} = k_{0e}$. At this point the system matrix will be proportional to Jordan block form as

$$\underline{\mathbf{M}}|_{k_0 = k_{x0}} = \underline{\mathbf{M}}_e = -\omega\mu \left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right). \tag{9}$$

It is worth mentioning that recently certain EPDs have been descfibed also be described using bifurcation theory [4], [5], where we can find that $D(k) = 0$ and $dD(k)/dk = 0$ at the degeneracy condition $k_0 = k_{0e}$. Thus, EPD are considered branch-point singularities in the complex-frequency plane.

The general solution of (4) with an initial condition $\Psi_{zo} = [\ E_{yo} \quad H_{xo}\ ]^T$ at $z = 0$ is

$$\mathbf{\Psi}(z) = \exp(-j\underline{\mathbf{M}}z)\mathbf{\Psi}_{z0}. \tag{10}$$

Using Taylor series expansion of exponential function and using the fact that $\underline{\mathbf{M}}_e^n = \mathbf{0}, \quad \forall\ n \geq 2$, the exponential matrix function at EPD can be expanded as

$$\exp(-j\underline{\mathbf{M}}_e z) = \sum_{n=0}^{\infty} \frac{(-j\underline{\mathbf{M}}_e z)^n}{n!} = \underline{\mathbf{I}} - j\underline{\mathbf{M}}_e z \tag{11}$$

Substituting (11) in (10), the general solution of (4) at EPD is expressed as

$$\mathbf{\Psi}(z) = E_{y0} \left(\begin{array}{c} 1 \\ 0 \end{array} \right) + H_{x0} \left(\begin{array}{c} j\omega\mu z \\ 1 \end{array} \right) \tag{12}$$

Substituting (12) into (2), the transverse fields at EPD is finally expressed as

$$\begin{array}{c} E_y(x,y,z) = (E_{y0} + j\omega\mu H_{x0} z)\sin(k_{x0}x) \\ H_x(x,y,z) = H_{x0}\sin(k_{x0}x) \end{array} \tag{13}$$

Considering a numerical example pertaining to a rectangula waveguide with $a = 17$mm and $b = 7$ mm. The evolution of the eigenvalues and eigenvectors with respect to frequency is shown in Fig. 2. It is obvious that at the EPD the waveguide has the same eignevalues and eigenvectors. The evolution of the electric field distribution along the waveguide is shown in Fig. 3 assuming an initial fields $E_{y0} = 10$ V/m and $H_{x0} = 0.02$ A/m. The figure shows how electric field evolves with a linear growth with z as we approach the EPD condition.

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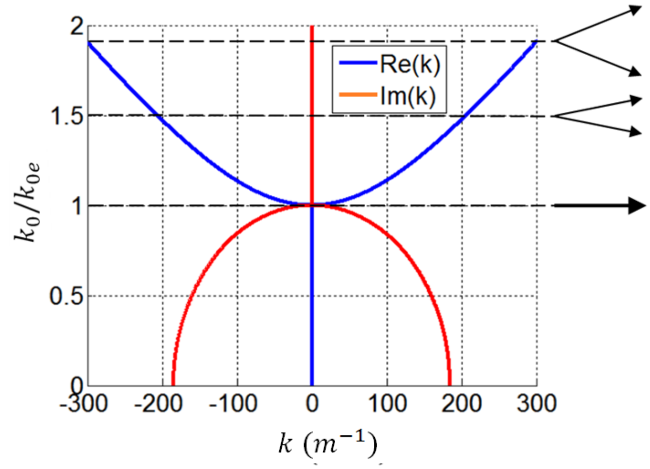

Fig. 2: Evolution of the eigenvalues and eigenvectors representing the dominant TE_{10} mode of a uniform rectangular waveguide.

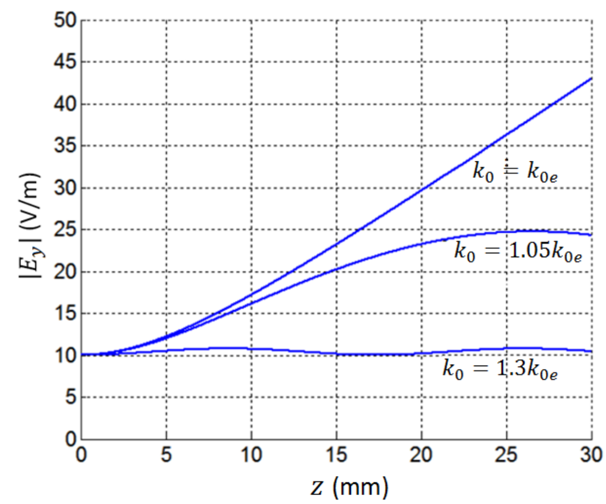

Fig. 3: Transverse electric field of the the dominant mode in a uniform rectangular waveguide calculated at $x = a/2$ when it has an initial fields $E_{y0} = 10$ V/m and $H_{x0} = 0.02$ A/m , at $z = 0$, at different operating frequencies.

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