Far-field Extrapolation of the Body-of-Revolution Parabolic Wave Equation

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***Abstract*—The parabolic wave equation (PWE) has been used extensively to model propagation in electrically large domains dominated by forward scatter. A PWE method for modeling propagation through an inhomogeneous body-of-revolution (BOR) was recently described, wherein a three-dimensional result was synthesized from a summation of independent solutions obtained with two-dimensional solvers. Far-field solutions were obtained by applying scalar diffraction theory on the axial boundary—using a cylindrical coordinate system—which required a numerical domain with a large radial extent. This paper presents an efficient method for extrapolating the field on the radial boundary into the far field, allowing for a numerical domain with a smaller radial extent, thereby increasing computational efficiency.**

I. Introduction

In [1] a BoR-PWE was presented for problems adhering to the geometry shown in Fig. 1. The BoR-PWE allowed for computationally efficient modeling of electrically large problems with weak backscatter under a paraxial approximation by seeking solutions of a scalar Helmholtz-type equation,

$$\left(\nabla^2 + k_0^2 \epsilon_{\mathrm{r}}(\rho, z)\right) \psi(\rho, z, \phi) = 0, \tag{1}$$

of the form

$$\psi(\rho, z, \phi) = \sum_{n=-N}^{N} \psi_n(\rho, z)\, e^{in\phi} \tag{2}$$

with $\psi_n(\rho, z) = e^{-ik_0 z} u_n(\rho, z)$. Under the assumed cylindrical symmetry of the relative permittivity, each u_n is regarded as the solution to a PWE under the Claerbout approximation [2]. This approach is numerically efficient because whereas (1) must be solved on the entire three-dimensional domain *simultaneously*, the BoR PWE entails solving $2N + 1$ independent two-dimensional problems, each of which can be solved by *sequentially* advancing the solution forward in the z direction.

The far-field extrapolation method in [1] required a numerical domain with a large enough radial extent to ensure that the radial boundary was not significantly illuminated. Here, simple expressions are derived for extrapolation of the field on the radial boundary, allowing a PWE domain with a much smaller radial extent to be employed. Because perfectly matched layers do not effectively suppress waves with grazing incidence [3], an absorbing layer is added. A non-local boundary condition [4] could serve the purpose of both layers, but its implementation is more involved. By reducing the radial extent of the numerical domain, far-field solutions can be obtained much more efficiently.

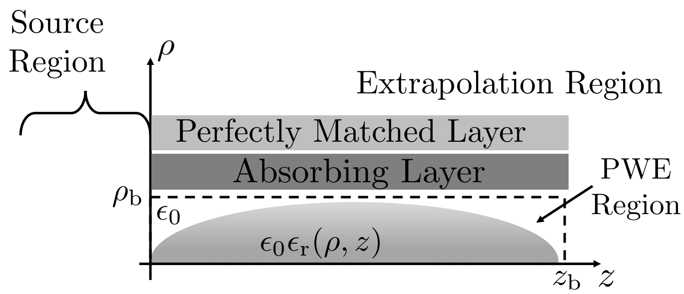


Fig. 1. Geometry for modeling wave propagation through an inhomogeneous BoR. The field in the extrapolation region is computed by extrapolating the field on ρ_{b} and z_{b} using Huygen's principle.

II. Derivation

Using Huygen's principle, the field outside of the PWE domain, due to the field on the radial boundary can be written as

$$\psi^{(\mathrm{cyl})}(\boldsymbol{r}) = \int_0^{z_{\mathrm{b}}} dz' \int_0^{2\pi} d\phi'\, \rho' \left(\psi(\boldsymbol{r}') \frac{\partial g_0(\boldsymbol{r}, \boldsymbol{r}')}{\partial \rho'} - g_0(\boldsymbol{r}, \boldsymbol{r}') \frac{\partial \psi(\boldsymbol{r}')}{\partial \rho'} \right) \Bigg|_{\rho' = \rho_{\mathrm{b}}}, \tag{3}$$

with

$$g_0(\boldsymbol{r}, \boldsymbol{r}') = -\frac{e^{-ik_0|\boldsymbol{r} - \boldsymbol{r}'|}}{4\pi|\boldsymbol{r} - \boldsymbol{r}'|}. \tag{4}$$

Substituting (2) and

$$g_0(\boldsymbol{r}, \boldsymbol{r}') \sim -\frac{e^{-ik_0 r}}{4\pi r}\, e^{ik_0 \rho' \sin\theta \cos(\phi - \phi')}\, e^{ik_0 z' \cos\theta}, \tag{5}$$

into (3) where $\sim$ denotes far-field equality and θ is the polar angle, results in

$$\psi^{(\mathrm{cyl})}(\boldsymbol{r}) \sim \frac{e^{-ik_0 r}}{r} \sum_{n=-N}^{N} e^{in\phi}\, f_n^{(\mathrm{cyl})}(\theta), \tag{6}$$

where

$$f_n^{(\text{cyl})}(\theta) = i^n \frac{\rho_\text{b}}{2} \left[k_\rho \, J_n'(k_\rho \rho_\text{b}) \int_0^{z_\text{b}} dz' \, \psi_n(\rho_\text{b}, z') \, e^{ik_z z'} + J_n(k_\rho \rho_\text{b}) \int_0^{z_\text{b}} dz' \frac{\partial \psi_n(\rho_\text{b}, z')}{\partial \rho_\text{b}} \, e^{ik_z z'} \right]. \tag{7}$$

In (7), J_n denotes the Bessel function of the first kind and order n [5], J_n' is the derivative of J_n with respect to its argument, $k_\rho = k_0 \sin\theta$, and $k_z = k_0 \cos\theta$. Numerically, $\partial\psi_n/\partial\rho'$ is evaluated with a central-difference approximation.

The field on the axial boundary is similarly extrapolated as

$$\psi^{(cap)}(\boldsymbol{r}) \sim \frac{e^{-ik_0 r}}{r} \sum_{n=-N}^{N} e^{in\phi} \, f_n^{(\text{cap})}(\theta), \tag{8}$$

where

$$f_n^{(\text{cap})}(\theta) = i^{(n+1)} k_0 \cos\theta \int_0^{\rho_\text{b}} d\rho' \, \rho' J_n(k_\rho \, \rho) \, \psi_n(\rho', z_\text{b}) \, e^{ik_z z_\text{b}}. \tag{9}$$

The far-field pattern is thus defined as

$$f(\theta, \phi) = \sum_{n=-N}^{N} e^{in\phi} \left(f_n^{(\text{cyl})}(\theta) + f_n^{(\text{cap})}(\theta) \right). \tag{10}$$

III. Results and Conclusion

Fig. 2 shows an example wherein both the axial and radial boundary are appreciably illuminated. Fig. 3 shows the far-field patterns as a function of θ and ϕ when extrapolated directly from the $(z = 0)$-plane, compared to the result when extrapolated from both the axial and radial boundaries. The results are in very close agreement.

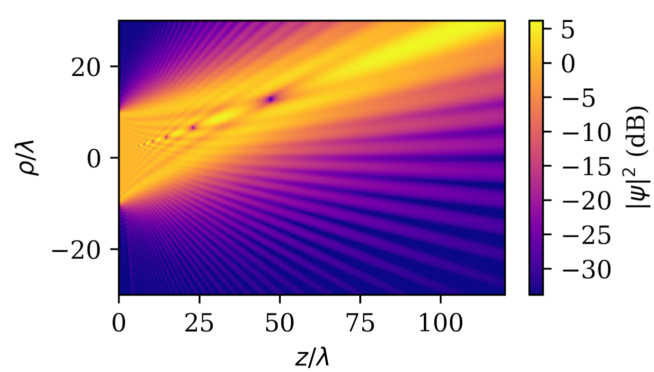
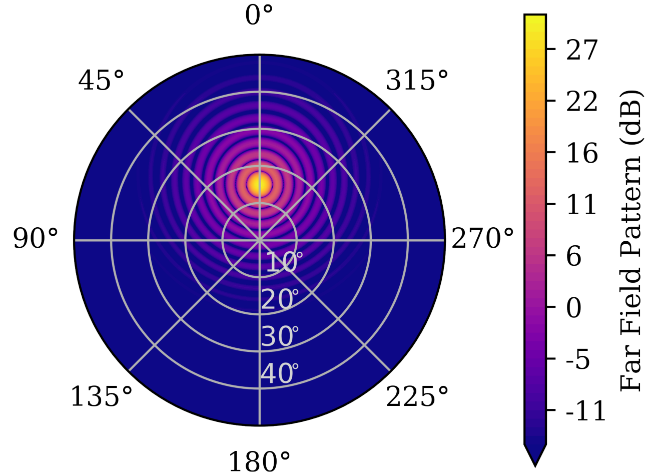
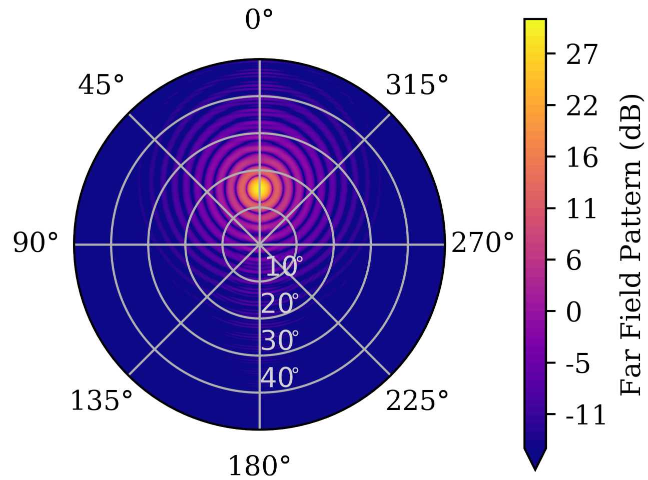


Fig. 2. BoR-PWE prediction of transmission loss from a 20-wavelength-diameter circular aperture illuminated by a planewave propagating at 15° relative to the z-axis.

These results confirm the accuracy of the cylindrical-boundary-extrapolation method, which will allow smaller numerical domains to be used in the PWE implementation.



(a)



(b)

Fig. 3. Far-field patterns extrapolating the field directly from the aperture (a), and using both the axial and radial boundaries (b). The radial coordinate corresponds to the polar angle, θ, and the azimuthal coordinate corresponds to ϕ.

References

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