Ultimate Intrinsic Signal-to-Noise Ratio of MRI Surface Coils for a Lossy Dielectric Elliptical Cylinder Model

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***Abstract*—The theoretical framework in MRI applications of signal-to-noise ratio (SNR) calculation is formulated for 1) SNR within a lossy elliptical cylinder excited by arbitary surface coils using dyadic Green functions and, 2) ideal current distribution corresponding to Ultimate Intrinsic SNR. The scattering problem of the infinite cylinder is solved by eigenfunction expansions.**

I. Introduction

MRI surface coils are widely useful to obtain images and spectra from tissues close to them for their high signal-to-noise ratio (SNR). In this work, regardless of imaging parameters, the intrinsic SNR of an arbitrary surface coil is calculated for an infinitely long homogeneous dielectric elliptical cylinder with losses. A general approach to computation of the intrinsic SNR using the reciprocity principal and the Poynting vector was developed in [1] for a circular cylinder case. We follow that approach to derive the intrinsic SNR with mode expansions of dyadic Green functions in elliptic cylindrical coordinates. Transformations between angular Mathieu functions with different complex parameter q is applied since the orthogonality relations of Mathieu angular functions are not valid across the boundary, where the medium property changes.

To obtain this theoretical best possible SNR independent of any particular coil geometry, i.e. the ultimate intrinsic SNR (UISNR) introduced in [2], we maximize the SNR expression by finding the optimum series expansion coefficients of a generic surface current distribution. An earlier work was done in [3] for a circular cylinder case.

II. Analytic solutions of SNR

A. Signal and Noise Calculation

For an arbitrarily located magnetic dipole $\mathbf{M}_1$ located at $\mathbf{r}'$ within the cylinder shown in Fig. 1 and a current source $\mathbf{J}_2$ outside the cylinder, by assuming a unit current flowing in the coil, we use the reciprocity principle to find the signal

$$-j\omega\mu_0\mathbf{M}_1(\mathbf{r}')\cdot\mathbf{H}_2(\mathbf{r}') = -j\omega\mu_0\int_S \mathbf{H}_1(\mathbf{r})\cdot \mathrm{d}\mathbf{S} \tag{1}$$

and the Poynting vector for power losses in the tissue

$$P_l = \frac{\sigma}{2}\int_V \mathbf{E}_2(\mathbf{r})\cdot\mathbf{E}_2^*(\mathbf{r})\,\mathrm{d}V \tag{2}$$

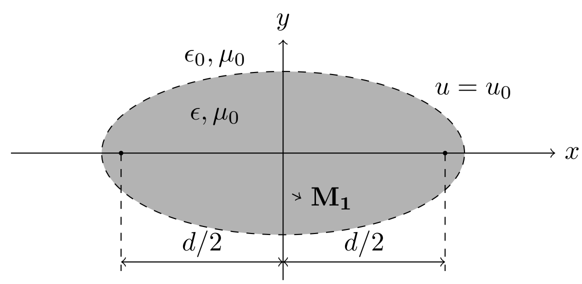

Figure 1. Cross section of the elliptical cylinder with a boundary at $u = u_0$

Thus, the SNR at any point within the cylinder is writen as

$$\mathrm{SNR}(\mathbf{r}') = \frac{|\omega\mu_0\mathbf{M}_1(\mathbf{r}')\cdot\mathbf{H}_2(\mathbf{r}')|}{\sqrt{8kT\Delta f P_l}} \tag{3}$$

where k is Boltzmann's constant, T is the tissue's absolute temperature, and Δf is the noise bandwidth.

B. Fields Calculation

The primary magnetic field set up by $\mathbf{M}_1$ in the region $u_s < u < u_0$ is found by introducing the dyadic Green function

$$\mathbf{H}_p(\mathbf{r}) = k^2\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}_s)\cdot\mathbf{M}_1(\mathbf{r}_s) = \sum_m \frac{jk^2}{2\pi(k^2-h^2)} \left(A_{\substack{e\\o}m}(u_s,v_s,h)\mathbf{M}^{(2)}_{\substack{e\\o}m}(q,h) + B_{\substack{e\\o}m}(u_s,v_s,h)\mathbf{N}^{(2)}_{\substack{e\\o}m}(q,h)\right) \tag{4}$$

where $\mathbf{M}^{(2)}_{\substack{e\\o}m}(q,h)$ and $\mathbf{N}^{(2)}_{\substack{e\\o}m}(q,h)$ are vector wave functions in terms of Mathieu functions in [4], h is the component of the wave vector k along $\hat{\mathbf{z}}$, $q = (k^2-h^2)d^2/8$ is the parameter of the Mathieu equations following the notation in [5], and $A_{\substack{e\\o}m}(u_s,v_s,h)$ and $B_{\substack{e\\o}m}(u_s,v_s,h)$ are known coefficients w.r.t. $\mathbf{M}_1(\mathbf{r}_s)$. We use the unknown coefficients, $C_{\substack{e\\o}m}$, $D_{\substack{e\\o}m}$, $c_{\substack{e\\o}m}$, and $d_{\substack{e\\o}m}$, to represent scattered fields $\mathbf{H}_s(u < u0)$ and outgoing fields $\mathbf{H}_1(u > u_0)$ as

$$\mathbf{H}_s(\mathbf{r}) = \sum_m \left(C_{\substack{e\\o}m}\mathbf{M}_{\substack{e\\o}m}(q,h) + D_{\substack{e\\o}m}\mathbf{N}_{\substack{e\\o}m}(q,h)\right) \tag{5a}$$

$$\mathbf{H}_1(\mathbf{r}) = \sum_m \left(c_{\substack{e\\o}m}\mathbf{M}^{(2)}_{\substack{e\\o}m}(q_0,h) + d_{\substack{e\\o}m}\mathbf{N}^{(2)}_{\substack{e\\o}m}(q_0,h)\right) \tag{5b}$$

The boundary conditions at $u = u_0$ consist of $\mathbf{H} \times \hat{\mathbf{n}}$ and $\mathbf{E} \times \hat{\mathbf{n}}$ in the Fourier domain, which decomposite into $\hat{\mathbf{v}}$ and $\hat{\mathbf{z}}$ components. Moreover, each component can be separate into $\cos nv$ and $\sin nv$ modes by converting angular Mathieu functions of parameter q_0 to ones of q based on [6] as

$$S_{\substack{e\\o}m}(q_0, v) = \sum_r{}' \alpha_{\substack{e\\o}m,r} S_{\substack{e\\o}r}(q, v) \tag{6a}$$

$$S'_{\substack{e\\o}m}(q_0, v) = \sum_r{}' \alpha_{\substack{e\\o}m,r} \sum_p{}' \beta_{\substack{o\\e}r,p} S_{\substack{o\\e}p}(q, v) \tag{6b}$$

where $\alpha_{\substack{e\\o}m,r}$ and $\beta_{\substack{o\\e}r,p}$ are known coefficients of the projection with a basis of Mathieu functions to avoid point-matching method. Thus, eight equations in total at $u = u_0$ are required to solve 8 unknown coefficients for each m. Four equations along $\hat{\mathbf{z}}$ can be rearranged to obtain the transmission/reflection relation as

$$\begin{aligned} C_{\substack{e\\o}p} = &\sum_m{}' \frac{(k_0^2 - h^2)\epsilon\alpha_{\substack{e\\o}m,p}}{(k^2 - h^2)k_0\epsilon_0} \frac{R^{(2)}_{\substack{e\\o}m}(q_0, u_0)}{R_{\substack{e\\o}p}(q, u_0)} c_{\substack{e\\o}m} \\ &- \frac{jk^2 A_{\substack{e\\o}p}}{2\pi(k^2 - h^2)} \frac{R^{(2)}_{\substack{e\\o}p}(q, u_0)}{R_{\substack{e\\o}p}(q, u_0)} \end{aligned} \tag{7a}$$

$$\begin{aligned} D_{\substack{e\\o}p} = &\sum_m{}' \frac{(k_0^2 - h^2)k\alpha_{\substack{e\\o}m,p}}{(k^2 - h^2)k_0} \frac{R^{(2)}_{\substack{e\\o}m}(q_0, u_0)}{R_{\substack{e\\o}p}(q, u_0)} d_{\substack{e\\o}m} \\ &- \frac{jk^2 B_{\substack{e\\o}p}}{2\pi(k^2 - h^2)} \frac{R^{(2)}_{\substack{e\\o}p}(q, u_0)}{R_{\substack{e\\o}p}(q, u_0)} \end{aligned} \tag{7b}$$

and substituted in the boundary conditions along $\hat{\mathbf{v}}$ as in (8) for each value of p.

Noticing that the coefficients of $\mathbf{H}_1$, $c_{\substack{e\\o}m}$ and $d_{\substack{o\\e}m}$, are linear combinations of $A_{\substack{e\\o}m}$ and $B_{\substack{o\\e}m}$, which are known coefficients given any dipole source $\mathbf{M}_1(\mathbf{r}')$, we can remove the dot product with $\mathbf{M}_1(\mathbf{r}')$ on both sides of (1) to find $\mathbf{H}_2$. Then the surface integral of $\mathbf{H}_1$ depends on the coil surface S.

III. Ideal Current Distribution for UISNR

To estimate the UISNR, the theoretical highest possible SNR regardless of coil geometry, we express the general SNR in the forms of eigenfunctions to find the coefficients of current distributions resulting in minimal power loss. Considering a surface current along the surface consisting of two parts, a divergence-free part and a curl-free part, we write its general form with Mathieu functions and weighting factors W_n as

$$\mathbf{K}_n(v, z) = \Big(W_n^M \nabla \times S^{(2)}_{\substack{e\\o}n}(q_0, v) + W_n^E \nabla S^{(2)}_{\substack{e\\o}n}(q_0, v)\Big)e^{jhz} \tag{9}$$

The fields inside the elliptical cylinder can be treated similarly as

$$\mathbf{H}_2(\mathbf{r}) = \sum_n \int \mathbf{M}_{\substack{e\\o}m}(q, h)V_n^M + \mathbf{N}_{\substack{e\\o}m}(q, h)V_n^N \, \mathrm{d}h \tag{10}$$

where V_n^M and V_n^N are combinations of expansion coefficients of the current desity pattern and boundary condition at the surface in (7) and (8). The ideal current pattern $\mathbf{W}_i$ yields the UISNR proportional to $\mathbf{W}_i^t\mathbf{S}/\mathbf{W}_i^t\mathbf{N}\mathbf{W}_i^*$, where $\mathbf{S}$ is the signal and $\mathbf{N}$ is noise covariance matrix from (1) and (2).

IV. Future Work

The next step will be deriving representations of optimal surface current distributions for specific cylindrical window coils around the lossy elliptical cylinder. A previous work for a sphere and circular cylinder was presented in [7]. Also, numerical results for optimized current pattern will be presented using ACM algorithm 934 in [6] to compute Mathieu functions for complex values of q.

References

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$$\begin{aligned} &\sum_r{}' \frac{jh}{k}\Big(\frac{jk^2}{2\pi(k^2 - h^2)} B_{\substack{o\\e}r} R^{(2)}_{\substack{o\\e}r}(q, u_0) + D_{\substack{o\\e}r} R_{\substack{o\\e}r}(q, u_0)\Big)\beta_{\substack{e\\o}r,p} - \frac{jk^2}{2\pi(k^2 - h^2)} A_{\substack{e\\o}p} R^{(2)\prime}_{\substack{e\\o}p}(q, u_0) - C_{\substack{e\\o}p} R'_{\substack{e\\o}p}(q, u_0) \\ &= \sum_m{}' \Big(-c_{\substack{e\\o}m} R^{(2)\prime}_{\substack{e\\o}m}(q_0, u_0)\alpha_{\substack{e\\o}m,p} + d_{\substack{o\\e}m} \frac{jh}{k_0} R^{(2)}_{\substack{o\\e}m}(q_0, u_0) \sum_r{}' \alpha_{\substack{o\\e}m,r}\beta_{\substack{e\\o}r,p}\Big) \end{aligned} \tag{8a}$$

$$\begin{aligned} &\sum_r{}' \frac{jh}{k}\Big(\frac{jk^2}{2\pi\epsilon(k^2 - h^2)} A_{\substack{e\\o}r} R^{(2)}_{\substack{e\\o}r}(q, u_0) + C_{\substack{e\\o}r} \frac{k}{\epsilon} R_{\substack{e\\o}r}(q, u_0)\Big)\beta_{\substack{o\\e}r,p} + \frac{k^2 h}{2\pi\epsilon(k^2 - h^2)} B_{\substack{o\\e}p} R^{(2)\prime}_{\substack{o\\e}p}(q, u_0) - D_{\substack{o\\e}p} \frac{jh}{\epsilon} R'_{\substack{o\\e}p}(q, u_0) \\ &= \sum_m{}' \Big(-\frac{k_0}{\epsilon_0} d_{\substack{o\\e}m} R^{(2)\prime}_{\substack{o\\e}m}(q_0, u_0)\alpha_{\substack{o\\e}m,p} + c_{\substack{e\\o}m} \frac{jh}{\epsilon_0} R^{(2)}_{\substack{e\\o}m}(q_0, u_0) \sum_r{}' \alpha_{\substack{e\\o}m,r}\beta_{\substack{o\\e}r,p}\Big) \end{aligned} \tag{8b}$$