Convergence Rates of 2D Moment Method Solutions for the MFIE and EFIE

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We give theoretical convergence rates for the surface current and backscattering amplitude from an infinite, PEC circular cylinder associated with typical moment method solutions to the magnetic and electric field integral equations. These results are then numerically compared to other scatterer geometries.

1. Introduction

While the method of moments is a popular method used to solve many EM scattering problems, understanding of its convergence is insufficient. Through Sobolev theory, the mathematics community has proven that standard numerical methods used in computational EM converge [1]. This paper adds to that work by providing simple error estimates for common moment method solutions which are less general than previous theoretical results but are given in terms of norms frequently used in engineering. The results are based on the authors' previous work in [2] and the studies of Warnick and Chew [3, 4].

The approach taken in this paper is to find analytical estimates for the relative RMS surface current error and relative scattering amplitude error for the case of an infinite, PEC circular cylinder, and then to compare the cylinder-based error estimates to numerically computed errors for other scatterer geometries, in order to determine if the error behavior of non-circular scatterers is qualitatively and quantititatively similar to that of the circular cylinder. We consider the electric field integral equation (EFIE) and the magnetic field integral equation (MFIE) for a TM-polarized incident plane wave.

2. MOM Error

For the method of moments, we employ pulse basis functions to expand the surface current on the scatterer and Dirac delta functions for testing the incident field on the scatterer. We consider both current error and scattering amplitude error. Current error is defined in the RMS sense, so that

$$\|\Delta J\|_{RMS} \triangleq \sqrt{\frac{1}{N}\sum_{m=1}^{N}|\Delta J_m|^2}$$

where N is the number of mesh nodes and ΔJ_m is the difference at the mth node between the exact surface current and the MoM surface current solution. Backscattering amplitude error is the absolute value of the difference of exact and MoM

	Exact Integration	Finite M	Flat-Facet
MFIE RMS Current Error	$0.9n_\lambda^{-2}$	$0.9n_\lambda^{-2}$	$1.5(ka)^{-1}n_\lambda^{-1}$
MFIE Backscattering Error	$1.5n_\lambda^{-2}$	$1.5n_\lambda^{-2}$	n_λ^{-1}
EFIE RMS Current Error	$2(ka)^{-\frac{1}{2}}n_\lambda^{-2}$	$(Mn_\lambda)^{-1}$	n_λ^{-2}
EFIE Backscattering Error	$1.9(ka)^{-1}n_\lambda^{-3}$	$1.4(Mn_\lambda)^{-1}$	$1.6(ka)^{-1}n_\lambda^{-2}$

Table 1: Theoretical relative error estimates for the method of moments applied to the problem of a TM-polarized plane wave incident on a circular PEC cylinder. Point testing and pulse expansion functions are used. $n_\lambda = \lambda/h$ is the mesh density, where h is the mesh element width. “Exact integration” denotes exact integration over the pulse expansion functions, with exact (curved facet) geometrical representatation. “Finite M” refers to the use of an M-point Euler quadrature rule to evaluate diagonal and off-diagonal moment matrix elements. “Flat-Facet” denotes a flat facet mesh with exact integration of moment matrix elements. Note that these error estimates break down near internal resonances of the cylinder. Error estimates valid at resonances are given in [2, 3].

solutions for the backscattering amplitude. It was shown in [2, 3] that the solution error can be expressed in the form $f(ka)n_\lambda^{-r}$, where $f(ka)$ is a function of the electrical size of the scatterer, r is the order of convergence, and n_λ is the number of mesh unknowns per wavelength, where $n_\lambda = \lambda/h$ and h is the mesh element width.

We analyze the following three sources of error: discretization error, quadrature error, and geometrical discretization error. It is found that these error contributions can be characterized by the difference in the eigenvalues of the moment matrix and the eigenvalues of the integral operator [3]. We consider only smooth scatterers. Error for non-smooth scatterers can be found in [5].

Discretization Error. Associated with any method of moments implementation is discretization error caused by representing the integral equation operator in a discrete basis. To analyze this source of error, we assume an exact geometrical representation of the scatterer and exact integration of moment matrix elements.

Quadrature Error. If moment matrix elements are evaluated with an M-point Euler quadrature rule, the MFIE solution error is only weakly sensitive to the number of quadrature points used. The exception to this rule is the case of a single point quadrature rule, where solution error decreases dramatically, becoming third order in n_λ. This is a special case that will likely not extend to a 3D MoM implementation.

Because of the singularity of the kernel, the EFIE is more sensitive to error introduced by a finite quadrature rule. It is found that EFIE convergence becomes first order in n_λ if the number of quadrature points does not increase proportionally to n_λ^2. (Special integration rules such as non-classical Gaussian quadrature and Duffy’s transform can decrease the number of required integration points.)

Geometrical Discretization Error It is a common practice to represent a curved scatterer with a flat-facet mesh. If this is done, additional error is introduced to the moment method solution. The MFIE is more sensitive to geometrical discretization error than the EFIE, as the convergence rate decreases to first order for a flat-facet mesh. This occurs because the first-order correction to the self-term of the MFIE from the integral part of the operator is curvature-dependent, and so cannot be used for a flat-facet mesh.

3. Extension to Other Geometries

In order to examine how error behaves for geometries other than the circular cylinder, we compute numerical errors for several smooth, non-circular scatterers, and compare these to the theoretical circular cylinder error estimates described above. The numerical error curves are generated by using a reference solution obtained using MoM on a high density mesh ($n_\lambda \approx 300$). The resulting computed error curves are given in Fig. 1 for the nine scatterers shown in Fig. 2. Scatterer 1 is a circular cylinder.

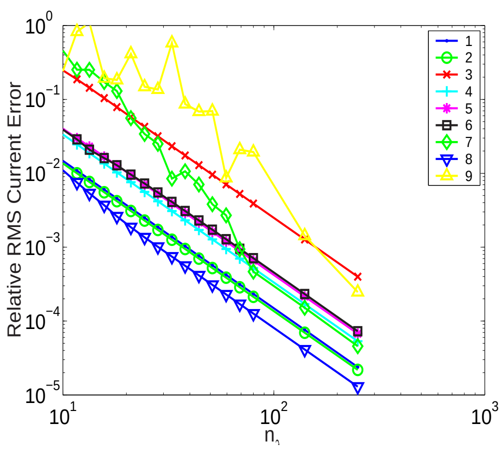

Figure 1: MFIE RMS current error vs. mesh unknowns per wavelength for non-circular scatterers, with exact integration and exact geometry. From the theoretical circular cylinder estimates in Table 1, convergence is expected to be second order.

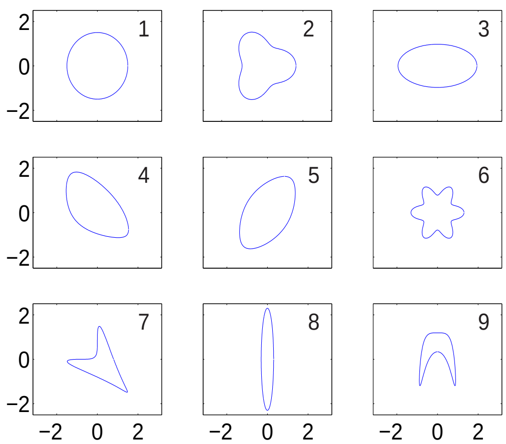

Figure 2: Scatterers compared in Fig 1. Though many of the scatterers have regions of high curvature, they are all smooth. Scatterer 1 is a circular cylinder. All scatterers have a perimeter of $3\pi\lambda$.

The error convergence rate is the same for all of the scatterers, whereas the absolute error constant is much larger in some cases. The largest initial errors are obtained for scatterers 7 and 9, which have corner regions of large curvature. These regions are close to edge singularities, which have been shown to decrease solution convergence rates [5].

4. Conclusions

We have provided error estimates for the surface current and scattering amplitude solutions to the MFIE and EFIE where the scatterer is a 2D circular cylinder. We analyzed three error sources: the error associated with representing the integral operator as the moment matrix, the error caused by a finite quadrature rule, and the error associated with a flat-facet geometrical discretization.

With second-order convergence as a baseline convergence rate, better accuracy (third order) is obtained for EFIE backscattering amplitude with exact geometrical representation and exact integration of moment matrix elements, and for the special case of MFIE with single point integration of off-diagonal elements and analytical diagonal elements. Worse error (first order) arises with the EFIE for a finite point quadrature rule for moment matrix integration, and with the MFIE for a flat-facet mesh. We have also demonstrated numerically that other non-circular scatterers have similar error convergence rates.

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